The Torelli map to $\mathcal{A}_g^{\mathrm{Vor}}$

Let y be a positive definite real quadratic form of $\mathbb{R}^g$ $\rightsquigarrow$ metric $\|x\|_y = \sqrt{x\, y\, {}^t x}$ $\forall x \in \mathbb{R}^g$

$\rightsquigarrow$ Delaunay cells : $\sigma = D(\overset{\in \mathbb{R}^g}{a_i})_{i \in I} = \{ \sum \lambda_i a_i : \sum \lambda_i = 1,\ \lambda_i \geq 0 \}$

s.t. $\exists\, \alpha \in \mathbb{R}^g$: 1) $\|a_i - \alpha\|_y = \min\limits_{\xi \in \mathbb{Z}^g} \|\xi - \alpha\|_y$

2) $\forall \xi \neq a_i$, $\|\xi - \alpha\|_y > \|a_i - \alpha\|_y$.

If y is a non-negative quadratic form of $\mathbb{R}^g$, $y = \begin{pmatrix} 0 & 0 \\ 0 & y'' \end{pmatrix}$, $y'' > 0$

$\rightsquigarrow$ pseudo-metric $\rightsquigarrow$ Delaunay cells $\mathbb{R}^{g'} \times \sigma''$ where σ'' is a D. cell w.r.t. y''. $g = g' + g''$.

$Y_g^+ = \{ y$ pos. def. real quad. form of $\mathbb{R}^g \}$ $\overline{Y}_g^+ = \{ y$ non-neg. real quad. form of $\mathbb{R}^g \}$

Def.

$y_1, y_2 \in \overline{Y}_g^+$, $y_1 \sim y_2$ $\Leftrightarrow$ Delaunay decompositions defined by y_1 and y_2 coincide

Def. A Delaunay-Voronoi cone (D-V cone) in $\overline{Y}_g^+$ is the closure $\Sigma = \Sigma(y)$ of a cone $\Sigma^\circ = \Sigma^\circ(y) = \{ y' \in \overline{Y}_g^+ : y' \sim y \}$.

Thm The D-V decomposition is admissible.

$$e: \mathbb{H}_g \longrightarrow \mathcal{T}_g^\circ \subset \mathcal{T}_g := \{ \tau : \tau = {}^t\tau \text{ w. nonzero coeff.} \}$$

$$\begin{array}{ccc} & & \shortparallel \\ \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & (\mathbb{C}^*)^{\frac{g(g+1)}{2}} \text{ coefficientwise mult.} \\ (z_{ij})_{ij} & \longmapsto & (\exp(2\pi i \cdot z_{ij}))_{ij} \end{array}$$

coordinate ring of $\mathcal{T}_g = \mathbb{C}[\ldots z_{ij}, z_{ij}^{-1} \ldots]$

D-V cone $\Sigma \rightsquigarrow \mathcal{X}_\Sigma := \operatorname{Spec}\left(\mathbb{C}[\prod_{i \leq j} z_{ij}^{A_{ij}}]_{A \in \overset{\vee}{\Sigma} \cap \overset{\vee}{Y}_{g,\mathbb{Z}}} \right)$ where $\overset{\vee}{Y}_{g,\mathbb{Z}} := \{ A \in \overset{\vee}{Y}_g : \langle A, y \rangle \in \mathbb{Z} \ \forall y \in Y_g \}$

D-V dec admissible $\Rightarrow$ The X_Σ can be patched together to form a torus embedding X of $\mathcal{T}_g$

X° = interior of the closure of $\mathcal{T}^\circ$ in X

$g = g' + g''$ $\quad$ $e_{g''} : \mathbb{H}_g \longrightarrow \mathcal{T}^{\circ}_{g',g''} \subset \mathcal{T}_{g',g''} := \mathbb{H}_{g'} \times \mathbb{Z}_{g',g''} \times \mathcal{T}_{g''}$
$\quad\; \vee\!\vee \;\; \vee\!\vee$
$\quad\; 0 \;\;\;\; 0$

$$\begin{pmatrix} \tau' & \tau''' \\ {}^t\tau''' & \tau'' \end{pmatrix} \mapsto (\tau', \tau''', e(\tau''))$$

$\mathbb{Z}_{g',g''}$ :: v. sp. of $g' \times g''$ matrices

$X_{g',g''} := \mathbb{H}_{g'} \times \mathbb{Z}_{g',g''} \times X_{g''}$, $X^{\circ}_{g',g''}$:= int. of the closure of $\mathcal{T}^{\circ}_{g',g''}$ in $X_{g',g''}$.

Locally $A_g^{V_{\Sigma}}$ is defined as $P_{g''} \backslash X^{\circ}_{g',g''}$

where $P_{g''} := \left\{ \begin{bmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & {}^tu^{-1} \end{bmatrix} \text{ where } \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in Sp(g', \mathbb{Z}) \text{ and } u \in GL(g'', \mathbb{Z}) \right\}$
$\quad\;\; \cap$
$Sp(g, \mathbb{Z})$

Let C be a stable curve of genus $g \geq 2$, let Γ be the dual graph of C. choose an orientation for Γ

$$0 \to H_1(\Gamma, \mathbb{Z}) \xrightarrow{i} X_e \xrightarrow{\partial} X_v \longrightarrow H_0(\Gamma, \mathbb{Z}) \to 0$$

where $\partial\left(\underset{v_i}{\bullet} \xrightarrow{e_i} \underset{u_i}{\bullet}\right) = u_i - v_i$. Choose a free basis $H_1(\Gamma, \mathbb{Z}) \simeq \mathbb{Z}^{g''}$ and express $i : \mathbb{Z}^{g''} \to \mathbb{Z}^d = X_e$ as a $g'' \times d$ matrix $A = ({}^t\alpha''_1, \dots, {}^t\alpha''_d)$
$\alpha''_i \in \mathbb{Z}^{g''}$.

Consider the Delaunay dec. on $X_e \otimes \mathbb{R}$ w.r.t. $\mathbb{1}_{X_e}$. Induce it on $H_1(\Gamma, \mathbb{Z}) \otimes \mathbb{R}$

<u>Thm</u>(*) (Namikawa) The induced dec on $H_1(\Gamma, \mathbb{Z}) \otimes \mathbb{R} \simeq \mathbb{Z}^{g''} \otimes \mathbb{R} = \mathbb{R}^{g''}$ is the Delaunay dec. of $\mathbb{R}^{g''}$ w.r.t. $\sum_{i=1}^{d} l_i \, {}^t\alpha''_i \alpha''_i$, $l_i > 0$.

Let C be a stable curve of genus $g \geq 2$. Let $z_1, \dots, z_d$ be double pts of C

$\exists$ a universal flat deformation space $\begin{array}{c} Z \\ \downarrow \pi \\ \mathcal{U} \end{array}$ with $\pi^{-1}(u_0) = C$ for $u_0 \in \mathcal{U}$

For a sufficiently small $\mathcal{U}$. $\mathcal{U}$ has a system of coordinates $(t_1, \dots, t_N)$, $N = 3g-3$ with center u_0 s.t. locally at z_i, Z is isom. to $\{u_i v_i - t_i = 0\} \subset \{(u_i, v_i, t_1, \dots, t_N)\} = \mathbb{C}^{N+2}$ and $\pi(u_i, v_i, t_1, \dots, t_N) = (t_1, \dots, t_N)$.

$$D_i = \{(t) \in \mathcal{U} : t_i = 0\}, \quad D = \bigcup_{i=1}^{d} D_i$$

The Torelli map is defined in $\mathcal{U} - D$ as: $T : \mathcal{U} - D \longrightarrow \mathbb{H}_g$

$$t \longmapsto \sum_{i=1}^{d} \frac{\log t_i}{2\pi i} \, {}^t\alpha_i \alpha_i + S(t)$$

where $S(t)$ is holomorphic in $\mathcal{U}$

$\alpha_i = (0, \alpha_i'')$

Thm $* \Rightarrow$ $\exists$ a D-V cone containing $\sum \lambda_i \, {}^t\alpha_i'' \alpha_i''$, $\lambda_i \geq 0$

$\Leftarrow\!\!\!\Rightarrow$ The map $\mathcal{U} - D \xrightarrow{T} \mathbb{H}_g \xrightarrow{e} \Sigma^{\circ}_{g \cdot g''}$

CRITERION

extends to an hol. map $\mathcal{U} \longrightarrow X^{\circ}_{g \cdot g''}$

Cor (Mumford) The Torelli map $M_g \longrightarrow A_g$ extends to $\overline{M}_g \xrightarrow{\text{a hol. map}} A_g^{\mathrm{Vor}}$.